中国科学技术大学
University of Science and Technology of China

From Simple Theory to Comprehensive Cultivation

My naive opinion on Mathematical Analysis Learning

Siwei Luo

School of the Gifted Young, USTC. 华罗庚数学科技英才班

Dec 22 2024

luosw@mail.ustc.edu.cn

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Section I: Theory

Theory

Theory

1. From Convergence to Continuity

2. From Differential to Derivative

Introducing Theories:

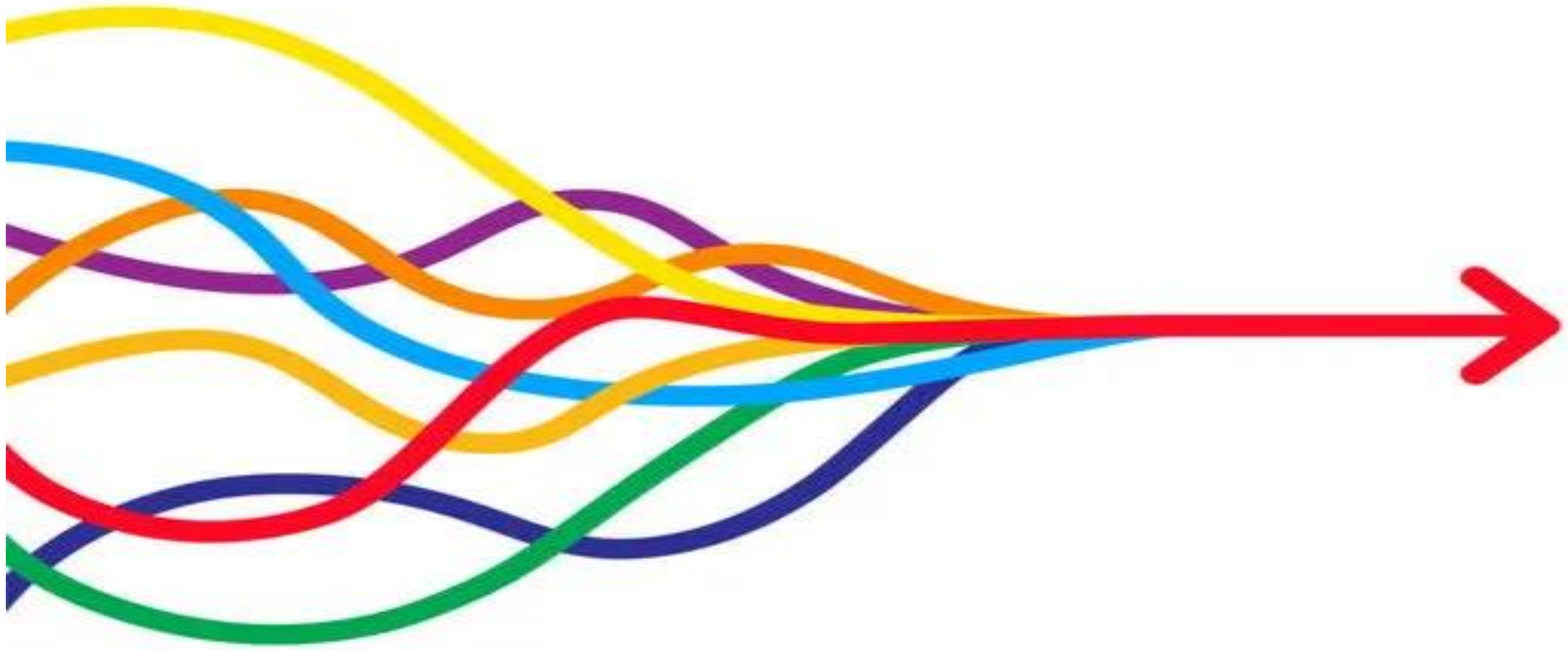
Relationship?

Origin of Idea?

1. From Convergence to Continuity

Definition (Convergence of Sequence)

The sequence (a_n) **converges** to a if for any $\varepsilon > 0$, there exists $N \in \mathbb{N}^*$, for any $n > N$, $|a_n - a| < \varepsilon$.



$$|a_n - a| < \varepsilon$$

?

“Neighborhood”?

1. From Convergence to Continuity

Definition (Limit of functions)

We say $\lim_{x \to x_0} f(x) = a$, if for any $\varepsilon > 0$, there exists $\delta > 0$, such that for any $|x - x_0| < \delta$, $|f(x) - a| < \varepsilon$ holds.

$|x - x_0| < \delta$ → $|f(x) - a| < \varepsilon_0$

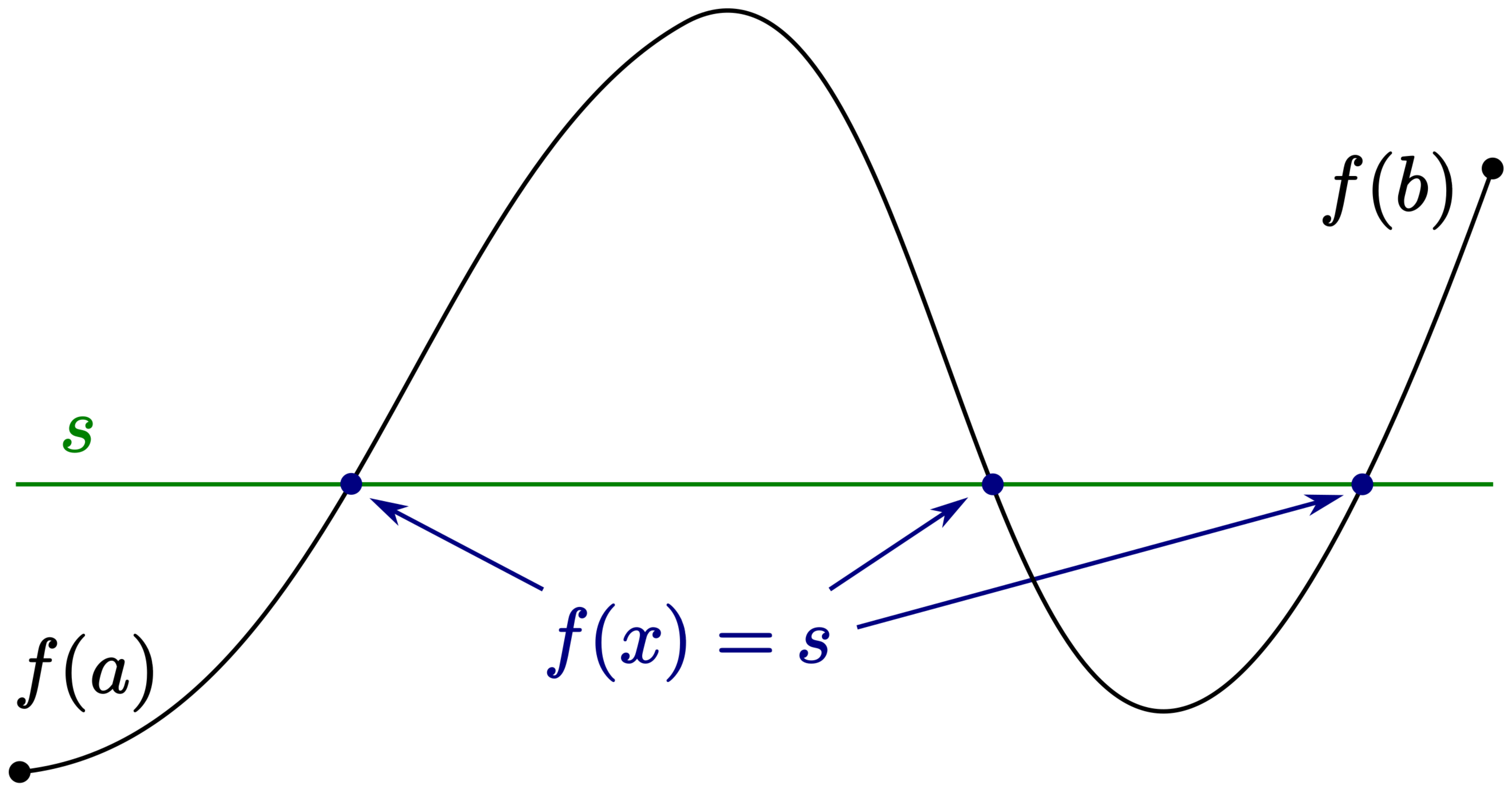
From Neighborhood

To Neighborhood

1. From Convergence to Continuity

Theorem (The Intermediate Value Theorem)

Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then for any value y between $f(a)$ and $f(b)$, there exists $c \in [a, b]$ s.t. $f(c) = y$.



connectedness of $[a, b]$

⇓

$f([a, b])$ connected?

1. From Convergence to Continuity

Theorem (The Extremal Value Theorem)

Let $f : [a, b] \to \mathbf{R}$ be a continuous function. Then f attains both a maximal value and a minimal value.

$f([a, b])$

How can it attach its extremal value?

Subset of $\mathbb{R}$ → **Compactness!**

1. From Convergence to Continuity

Theorem (Cantor's Theorem of Uniform Continuity)

If $f\colon [a, b] \to \mathbf{R}$ is a continuous function, then it is **unifomly continuous** on $[a, b]$.

Uniform Continuity → Continuity

Stronger

Inverse Implication ← Compactness!

2. From Differential to Derivative

Derivative

Differential

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Natural & Simple Definition

$$\mathrm{d}y = f'(x_0)\ \mathrm{d}x$$

What is the equation talking about??

2. From Differential to Derivative

Derivative ✗→ Differential

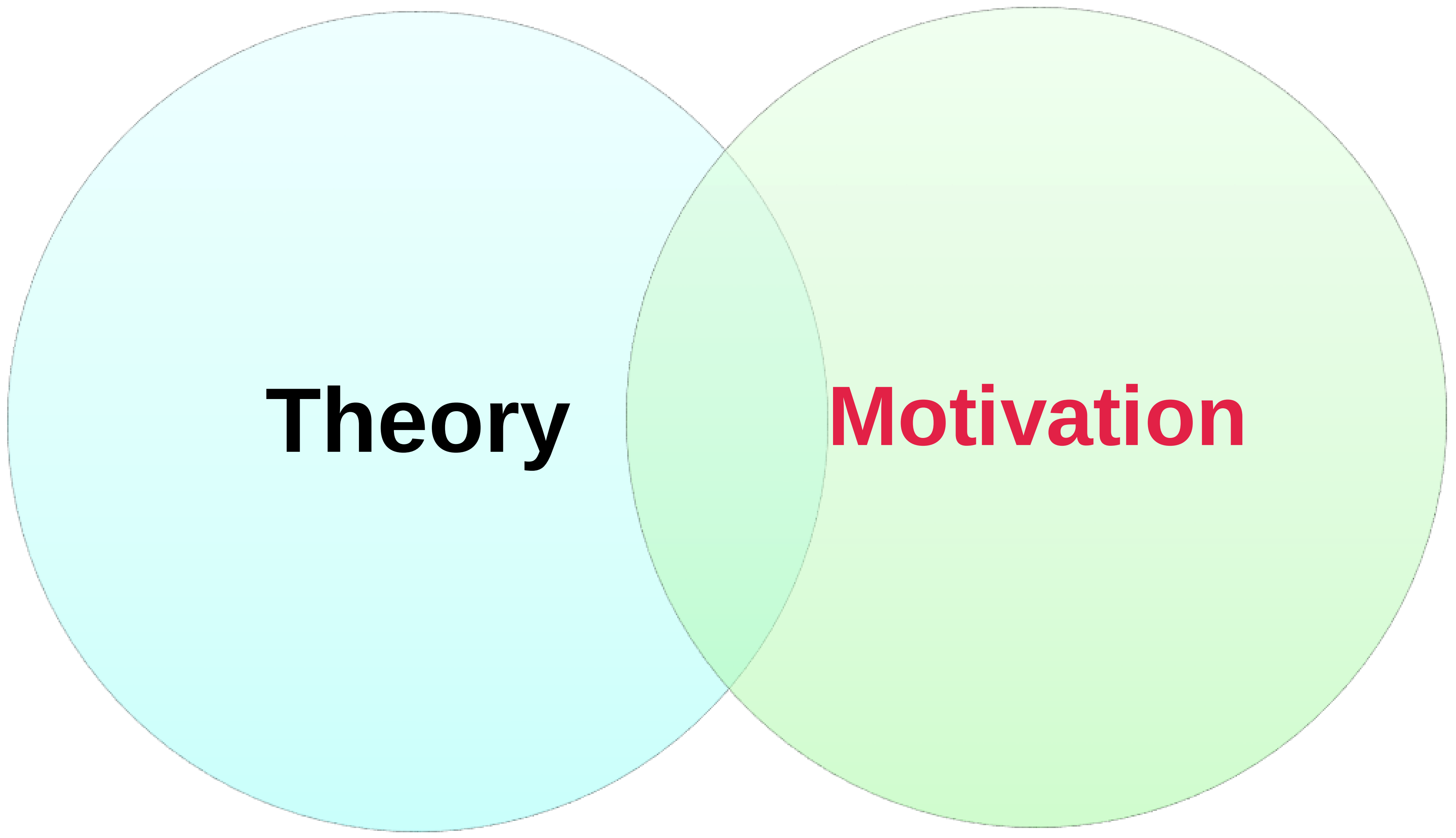
From Differential to Derivative

Differential → Derivative

PATH?

NECESSATY?

Section II: Motivation



Motivation

Topology?

Geometry?

1. From Convergence to Continuity

Neighborhood?

Convergence?

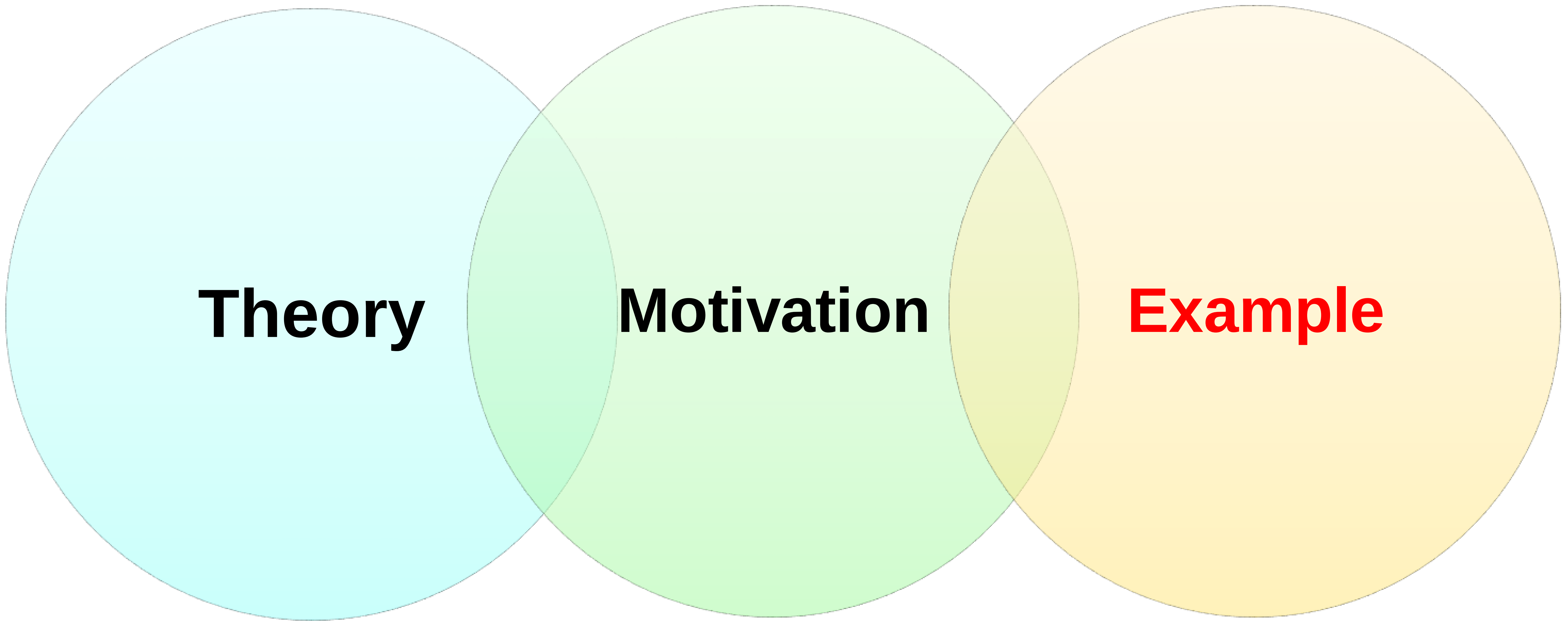
Compactness

2. From Differential to Derivative

$$\mathrm{d}y = f'(x_0)\ \mathrm{d}x$$

What is this?

Section III: Example



Example

1. From Convergence to Continuity

Neighborhood?

Converge ce?

Co

2. From Differe rivative

$\mathrm{d}y = f'(x_0)\ \mathrm{d}x$

What this?

What? Motivation.

WHY? EXAMPLE!

1. From Convergence to Continuity

Neighborhood?

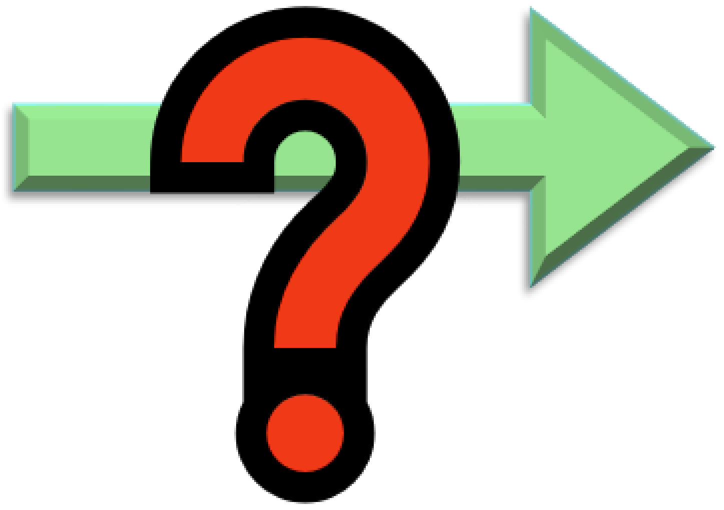
How to Define Concept Convergence

$$|a_n - a| < \varepsilon$$

$$|x - x_0| < \delta$$

$$|f(x) - a| < \varepsilon_0$$

From Neighborhood



To Neighborhood

Example?

“Preimage” of Neighborhood!

1. From Convergence to Continuity

Continuity

$$f^{-1}(N_{f(x_0)}) = N_{x_0}, \quad \forall N_{f(x_0)}$$

$$|x - x_0| < \delta$$
$$|f(x) - a| < \varepsilon_0$$

Example 1

Example 3

Example 2

Example ...

Observation

Gained Motivation

1. From Convergence to Continuity

Theorem (The Extremal Value Theorem)

Let $f : [a, b] \to \mathbf{R}$ be a continuous function. Then f attains both a maximal value and a minimal value.

Motivation — How can it attach its extremal value?

Example 1 — $[a, +\infty)$ → Bounded

Example 2 — $(a, b]$ → Has All Its Limit Point

1. From Convergence to Continuity

Examples → Bounded Closed $[a, b]$

$f([a, b])$

Motivation

$f^{-1}(N_{f(x_0)}) = N_{x_0}, \quad \forall N_{f(x_0)}$

Neighborhood → Open Intervals

Example 1 $[a, +\infty)$

Example 2 $(a, b]$

→ Compactness

1. From Convergence to Continuity

Definition (Compactness)

We say $A \subset \mathbf{R}$ is compact, if any its covering of open intervals $A \subset \bigcup_\alpha (a_\alpha, b_\alpha)$ admits a finite subcovering $A \subset \bigcup_{k=1}^{n} (a_{\alpha_k}, b_{\alpha_k})$.

Motivation

$[a, b]$ → $f([a, b])$

The Extremal Value Theorem

Compactness Preserved by Continuous Map

1. From Convergence to Continuity

Theorem (Cantor's Theorem of Uniform Continuity)

If $f\colon [a,b] \to \mathbf{R}$ is a continuous function, then it is **unifomly continuous** on $[a,b]$.

Motivation

$[a,b]$

Compactness?

Uniform Continuity

δ for all x

$\delta = \min_x \delta_x$

Finiteness of "x" chosen

1. From Convergence to Continuity

Consider some positive real number $\varepsilon > 0$. By continuity, for any point x in the domain M, there exists some positive real number $\delta_x > 0$ such that $d_N(f(x), f(y)) < \varepsilon/2$ when $d_M(x, y) < \delta_x$, i.e., a fact that y is within δ_x of x implies that $f(y)$ is within $\varepsilon/2$ of $f(x)$.

Let U_x be the open $\delta_x/2$-neighborhood of x, i.e. the set

$$U_x = \left\{ y \mid d_M(x, y) < \frac{1}{2}\delta_x \right\}.$$

Since each point x is contained in its own U_x, we find that the collection $\{U_x \mid x \in M\}$ is an open cover of M. Since M is compact, this cover has a finite subcover $\{U_{x_1}, U_{x_2}, \ldots, U_{x_n}\}$ where $x_1, x_2, \ldots, x_n \in M$. Each of these open sets has an associated radius $\delta_{x_i}/2$. Let us now define $\delta = \min_{1 \le i \le n} \delta_{x_i}/2$, i.e. the minimum radius of these open sets. Since we have a finite number of positive radii, this minimum δ is well-defined and positive. We now show that this δ works for the definition of uniform continuity.

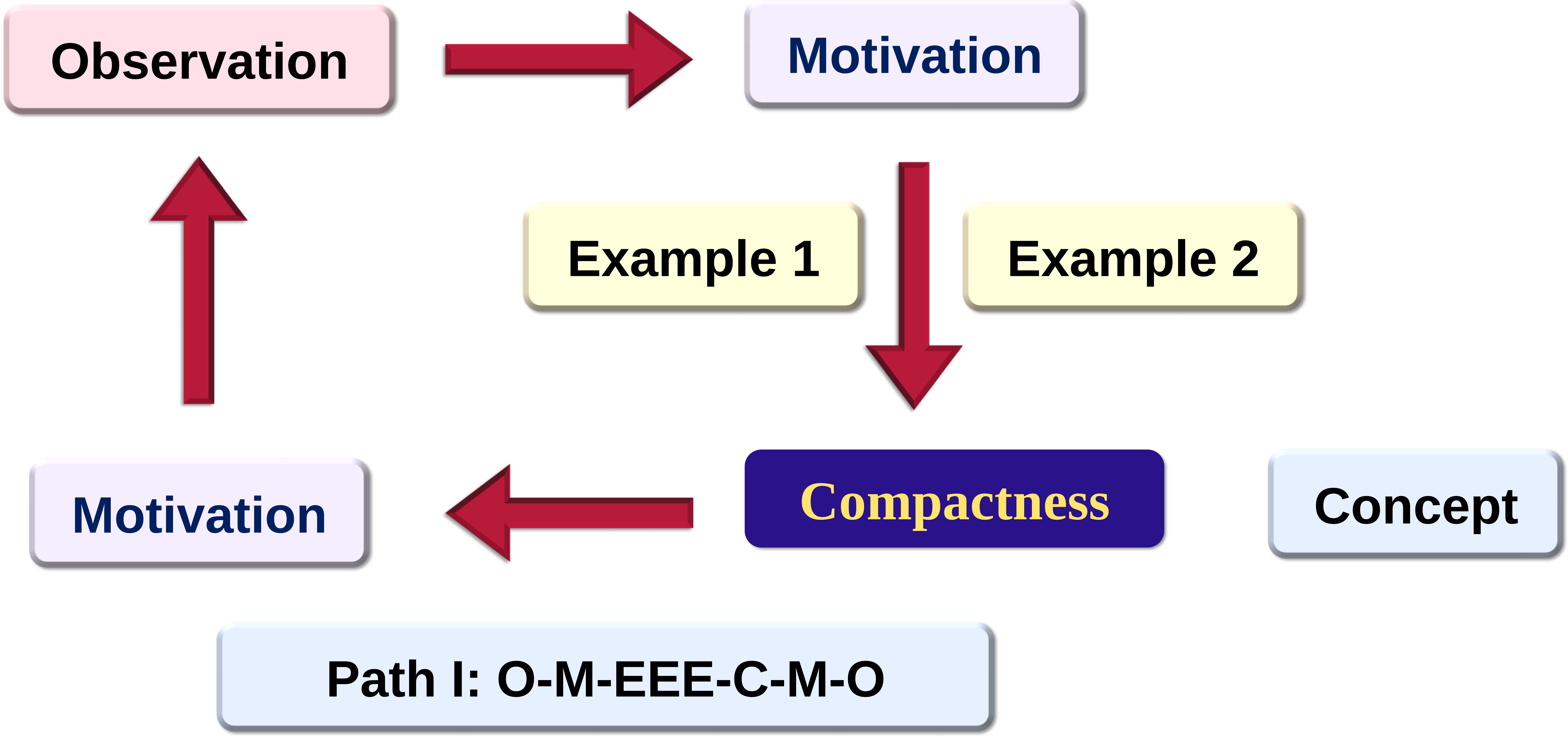
Suppose that $d_M(x, y) < \delta$ for any two x, y in M. Since the sets U_{x_i} form an open (sub)cover of our space M, we know that x must lie within one of them, say U_{x_i}. Then we have that $d_M(x, x_i) < \frac{1}{2}\delta_{x_i}$. The triangle inequality then implies that

$$d_M(x_i, y) \le d_M(x_i, x) + d_M(x, y) < \frac{1}{2}\delta_{x_i} + \delta \le \delta_{x_i},$$

implying that x and y are both at most δ_{x_i} away from x_i. By definition of δ_{x_i}, this implies that $d_N(f(x_i), f(x))$ and $d_N(f(x_i), f(y))$ are both less than $\varepsilon/2$. Applying the triangle inequality then yields the desired

$$d_N(f(x), f(y)) \le d_N(f(x_i), f(x)) + d_N(f(x_i), f(y)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

1. From Convergence to Continuity



2. From Differential to Derivative

Derivative

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Motivation

$$f(x_0 + h) - f(x_0) = f'(x_0)h + o(h)$$

$$\mathscr{A}(h) = f'(x_0)h$$

→ **Linear Transformation**

2. From Differential to Derivative

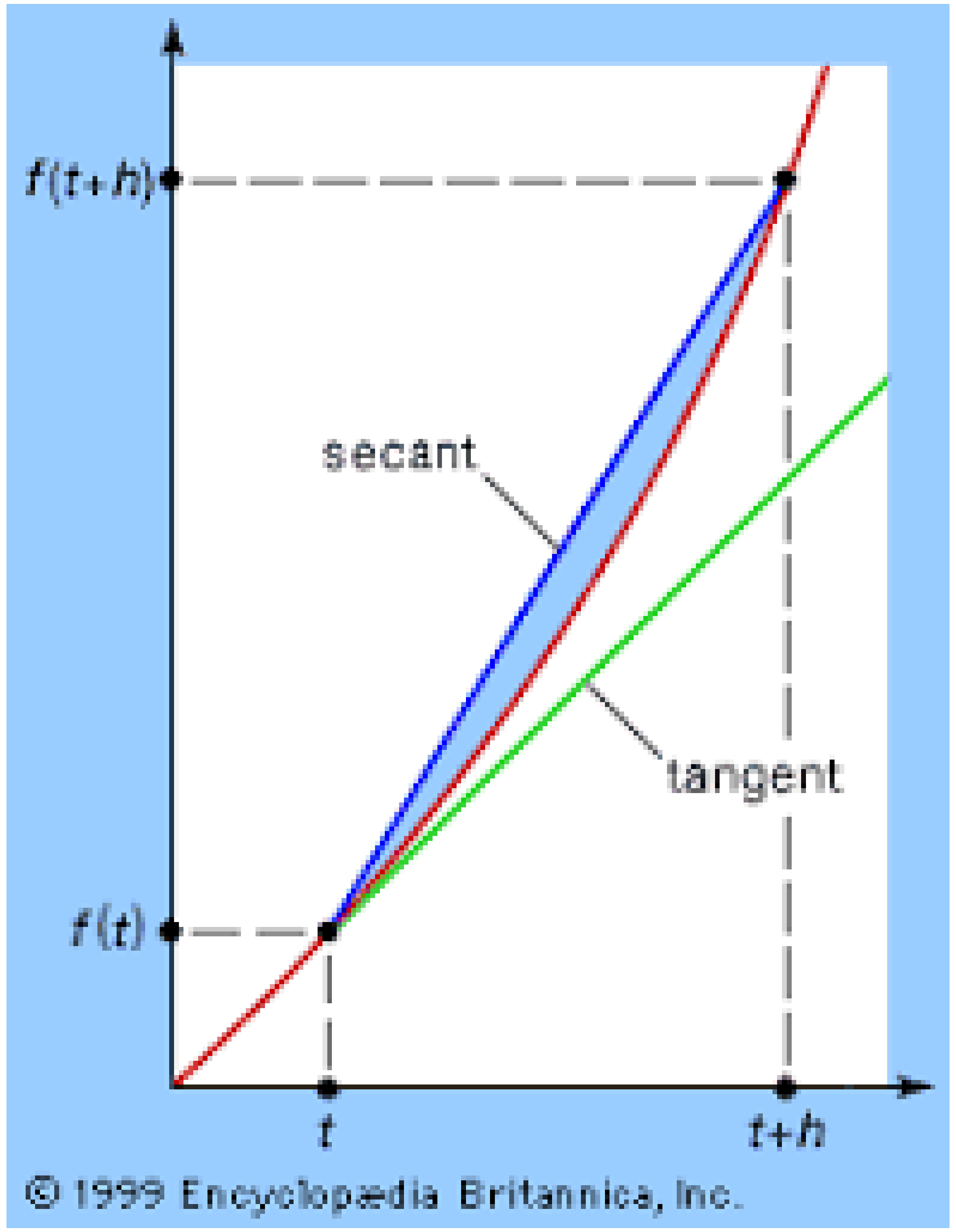
Motivation → $f(x_0+h) - f(x_0) = f'(x_0)h + o(h)$

$\mathscr{A} : \mathbf{R} \to \mathbf{R}$

$\mathscr{A}(h) = f'(x_0)h$

Linear Transformation

Use Linear Maps to Approach the Difference!



2. From Differential to Derivative

Motivation → **Use Linear Map to Approach the Difference!**

Definition (Differential)

Take $f : \mathbf{R}^n \to \mathbf{R}$ and $x^0 \in \mathbf{R}^n$. If there exists a linear map $\mathcal{A} : \mathbf{R}^n \to \mathbf{R}$ such that

$$f(x^0 + h) - f(x^0) = \mathcal{A}(h) + o(h),$$

then we call $\mathcal{A}$ the **differential** of f at x^0, denoted by $\mathrm{d}f_{x^0} = \mathcal{A} \in L(\mathbf{R}^n, \mathbf{R})$.

Motivation

What is the form of the linear map?

2. From Differential to Derivative

$$\mathcal{A}(h_1, \cdots, h_n) = a_1 h_1 + \cdots + a_n h_n$$

Motivation

What is the coefficients?

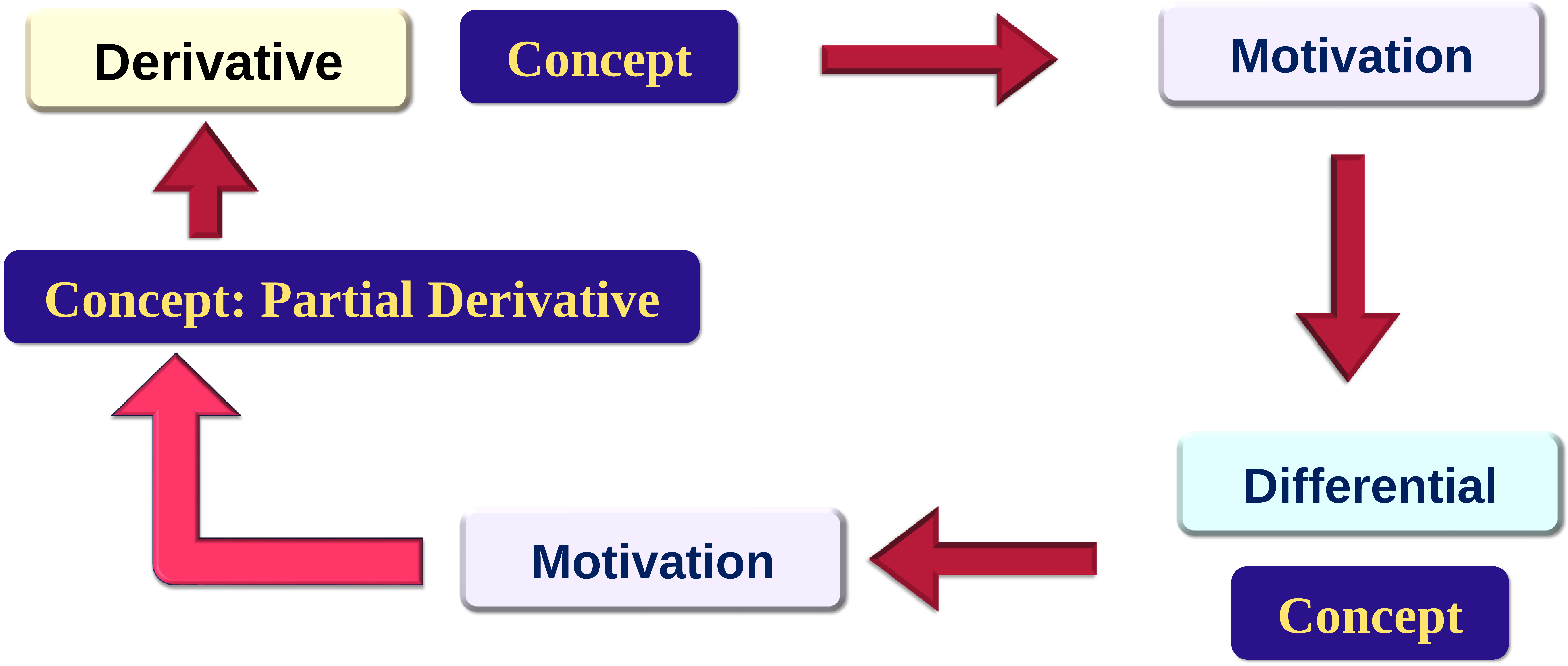
$$\frac{\partial f}{\partial x_k}$$

$$a_k = \frac{\mathcal{A}(he_k)}{h} = \lim_{h \to 0} \frac{f(x^0 + he_k) - f(x_0)}{h}$$

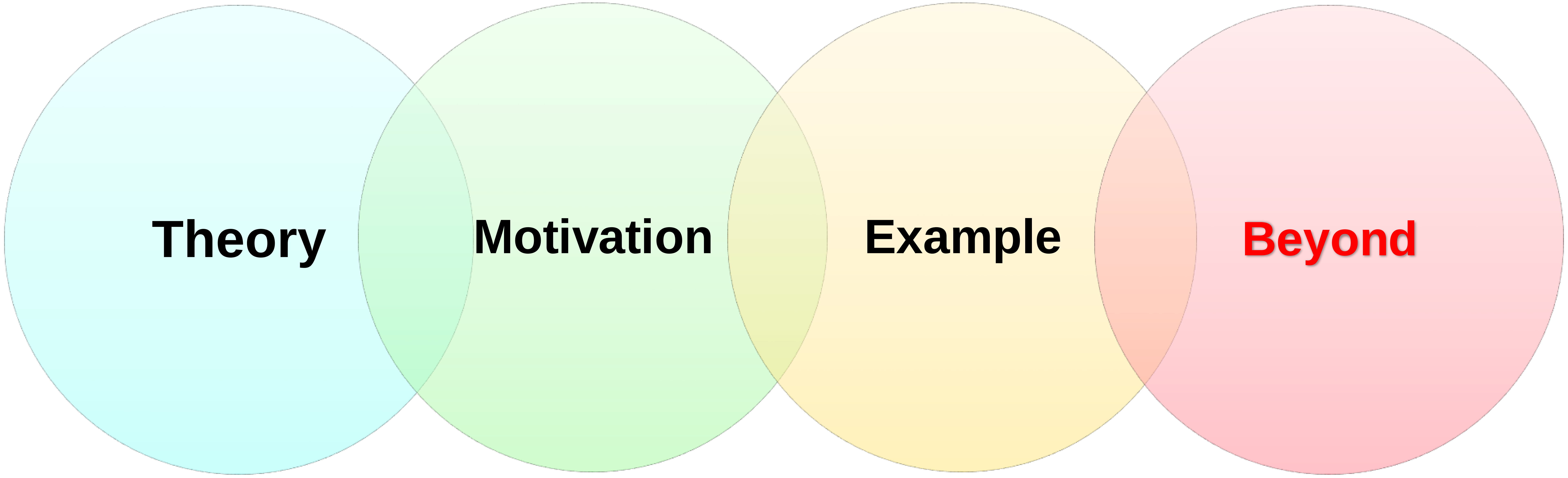
Motivation

Concept: Partial Derivative

2. From Differential to Derivative

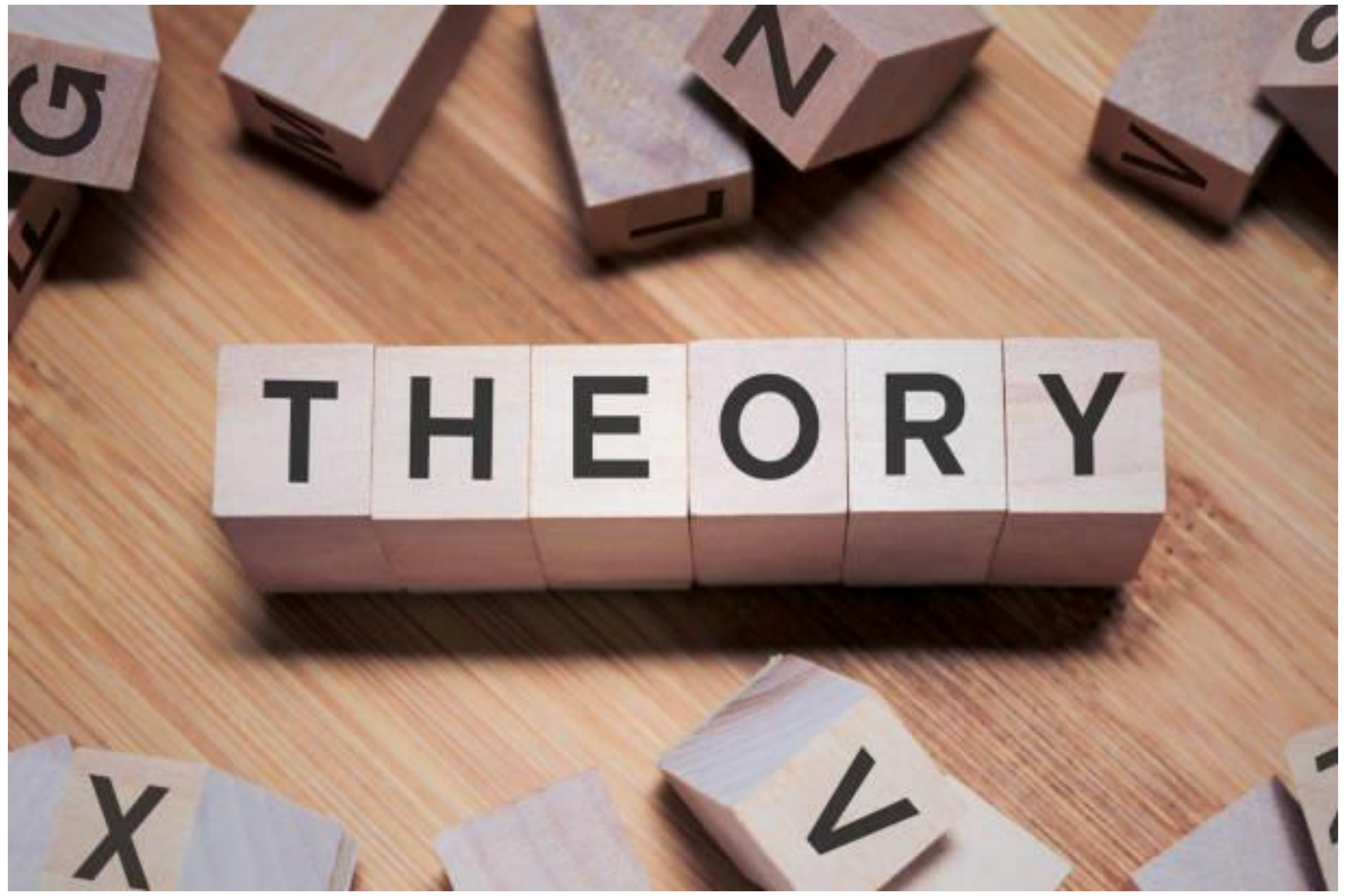


Section IV: Beyond (Mathematics)



Beyond Mathematics

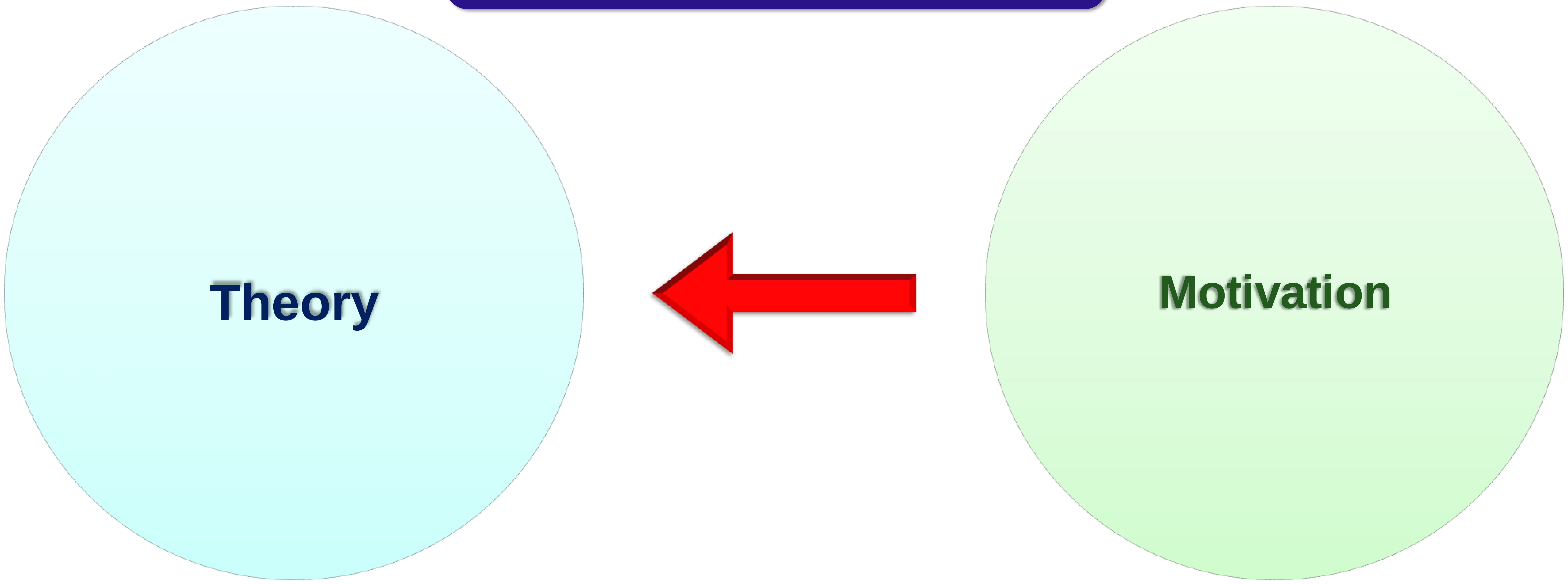
Get OUT of Abstract Theory



Skills & Literacy!

Similar Logic

Beyond Mathematics



- Our Opinions
- Difference is Normal!
- Experience
- Culture
- Education
- ……

Beyond Mathematics

Theory ← Example

Our Opinions

How to Act on Opinions?

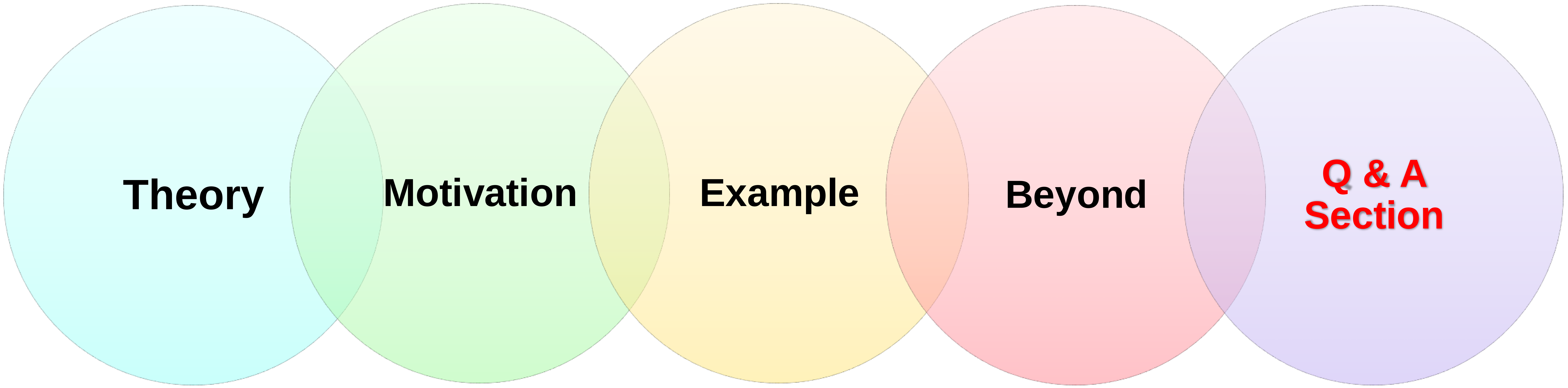
Special Cases

Argument

Counterexampl
e

Hearsay……

Section V: Q&A Section



Q & A
Section

Question by you & Answer of mine

Or

My Question for You First of All!

Nicolas Bourbaki

In Mathematics We Study…

Sets ← **Structure**

Maps

$$A \xrightarrow{f} B$$

Preserving The Structure

What is the case in Mathematical Analysis?

University of Science and Technology of China

Thank you!

Siwei Luo

School of the Gifted Young, USTC.　华罗庚数学科技英才班

Dec 22 2024　　luosw@mail.ustc.edu.cn

Luosw

QQ: 2986497980

Q & A
Section

Real for the time

Question by you & Answer of mine

Finite Gifts