Representation Theory Seminar 1 LECTURE NOTES

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Note 0.1 This is only an **outline** of the lecture, which is to say that there are no detailed explanation or proof in the note, which will be completed at the lecture.

1. Basic Concepts of Representation & Preliminaries

1.1. Representation and Examples

Informally, a representation of a group is a collection of invertible linear transformations of a $\mathbb{C}$-linear space that multiply together in the same way the group elements in G behave. (It is just like G acts on a linear space V.)

Let G denote a finite group and V be a $\mathbb{C}$-linear space.

Definition 1.1 (linear representation) A ***linear representation*** of G is a group homomorphism

$$\rho : G \to GL(V),$$

which often be denoted by (V, ρ).

Now here is some explanation of the "odd" definition. The notation $GL(V)$ is denoted by the group all the linear transformation of V i.e. the general linear group. The group homomorphism sends $g \in G$ to a linear transformation $\rho(g) : V \to V$. Try to answer the following questions by yourself:

- What is the significance that ρ must be a group **homomorphism**, not only a map?
- How to understand the first paragraph of this section?

Hence you can get a picture of the definition of the representation.

If there exists a basis B of V, we can get a group homomorphism $G \to GL_n(\mathbb{C})$, where $n = \dim_{\mathbb{C}} V$. The dimension of V, i.e. n, is called the **degree** of the representation, denoted by $\deg \rho := n$. Additionally, if ρ is injective, i.e. $\ker \rho$ is trivial, then we say the representation is **faithful**. Now we can consider the following examples of representations, whose properties we will pay attention to later.

Example 1.2 (trivial representation) Take $V = \mathbb{C}$ and $\dim V = 1$. Hence $GL(\mathbb{C}) = \mathbb{C}^{\times}$ and the representation is given by $\rho : G \to \mathbb{C}^{\times}$. Take $\rho(g) = 1$ for all $g \in G$ to be the identity on V. Hence we get the **trivial representation**. It is really important in the following theory, but so simple that it presents nothing but abstract.

Example 1.3 (permutation representation) Let G acts on a finite set X. Let V be a $\mathbb{C}$ −linear space with dimension $|X|$ and basis $\{e_x \mid x \in X\}$. Take $ge_x = e_{gx}$, then (V, ρ) is a representation. This representation is direct from the group acts and is called the **permutation representation**.

Why the representation is called the permutation representation? It can be explained since it is induced by a group action $G \to S_n$, where S_n is the permutation group of order n, and the matrix of each $g \in G$ is the permutation matrix.

Example 1.4 (regular representation) Take G as a basis of a linear space and let $g.g' = gg'$, then it forms a representation, which is called the **regular representation** with degree $\deg \rho = |G|$. Regular representation is a special case of permutation representation.

Now let us pay attention to some paticular interesting cases — 3 different representations of the group S_3. First of all, it has the trivial representation.

Example 1.5 (sign representation) Take $\rho : S_n \to \{-1, 1\}$ be the sign map of permutation. Hence we get the **sign representation** of S_n.

Example 1.6 Replace $\mathbb{C}$ by $\mathbb{R}$. Take $V = \mathbb{R}^2$ and $G = S_3$, which is isisomorphic to the group of symmetries of an equilateral triangle. The symmetries are the three reflections in the lines that bisect the equilateral triangle, together with three rotations. Positioning the center of the triangle at the origin of V and labeling the three vertices of the triangle as 1, 2, and 3, we then get a representation.

Note 1.7 In fact, **Example 1.6** is also a $\mathbb{C}$ −representation of G.

Note 1.8 We can also regard representation of a group acting on a linear space.

One important way to discover the properties of representations is the correspondence between group representations and group algebras.

Definition 1.9 (group algebra) We define the **group algebra**

$$\mathbb{C}[G] = \left\{ \sum_{g \in G} a_g g : a_g \in \mathbb{C} \right\},$$

as a linear space over $\mathbb{C}$ and the multiplication is given by

$$\left(\sum_{g \in G} a_g g \right) \left(\sum_{h \in G} b_h h \right) := \sum_{k \in G} \left(\sum_{gh = k} a_g b_h \right) k.$$

Example 1.10 Take the notation $k[G]$ as in **Definition 1.9** and take $G = \mathbb{Z}_4$, generated by s. Hence $k[G] \simeq k[x]/(x^4 - 1)$ as k −algebra.

Having defined the group algebra, now we can reveal the correspondence between group algebra and representations. Before that we need the definition of $\mathbb{C}[G]$ −module. The definition will be completed in the lecture.

Theorem 1.11 A representation of G over $\mathbb{C}$ has the structure of a $\mathbb{C}[G]$ −module. Conversely, every unital $\mathbb{C}[G]$ −module provides a representation of G over $\mathbb{C}$.

Is there any relationship between the group algebra and the regular representations? In fact, $\mathbb{C}[G]$ itself has a $\mathbb{C}[G]$ −module structure.

Regarding representations of G as $\mathbb{C}[G]$ −modules has the advantage that many definitions can be borrowed from module theory. The concepts, such as **subrepresentation, morphism of representations, direct sum of representations** can all be described by the relevant concepts in modules. Hence a little module theory is needed.

1.2. Commutative Algebra Preliminaries

We will use *Commutative Algebra* by Atiyah & Macdonald as textbook for this section to introduce basic module theory that we need.

In particular, we will cover the following sections:

- 2.1 modules and module homomorphisms
- 2.2 submodules and quotient modules
- 2.4 direct sum and product
- 2.8 restriction and extension on scalars
- 2.11 algebras

1.3. Subrepresentations & Morphism

Now we will regard a representation of G as a $\mathbb{C}[G]$ −module since **Theorem 1.11** holds.

To introduce the concept **subrepresentation**, we only need to pay attention to $\mathbb{C}[G]$ −submodule of a $\mathbb{C}[G]$ −submodule V. To specify a $\mathbb{C}[G]$ −submodule of V, it is necessary to specify an R −submodule W of V that is closed under the action of $\mathbb{C}[G]$, i.e. W is an invariant subspace of $\rho(g)$ for any $g \in G$. To be precise,

Definition 1.12 (subrepresentation) Let (V, ρ) be a representation of G and W be a subspace of V such that W is invariant under $\rho(g)$, for any $g \in G$. Hence we can get $\rho_W : g \to GL(W)$ and (ρ_W, W) is a **subrepresentation** of (V, ρ).

We make use of the notions of a homomorphism and an isomorphism of $\mathbb{C}[G]$ −modules. Since $\mathbb{C}[G]$ has as a basis the elements of G, to check that an $\mathbb{C}$ −linear homomorphism $f : V \to W$ is in fact a homomorphism of $\mathbb{C}[G]$ −modules, it suffices to check that $f(gv) = gf(v)$ for all $g \in G$ —— we do not need to check for every $x \in \mathbb{C}[G]$.

By means of the identification of $\mathbb{C}[G]$ −modules with representations of G (in **Theorem 1.11**) we may refer to homomorphisms and isomorphisms of group representations. In many books the algebraic condition on the representations that these notions entail is written out explicitly, and two representations that are isomorphic a real so said to be equivalent.

Given two $\mathbb{C}[G]$ −modules V and W, we may form their **direct sum** $V \oplus W$. We write $U = V \oplus W$ to mean that U has $\mathbb{C}[G]$ −submodules such that $U = V + W$ and $U \cap W = \{0\}$. In this situation,we also say that V and W are **direct summands** of U.

2. Irreducible & Completely Reducible Representation

2.1. Maschke's Theorem

We come now to our first nontrivial result, one that is fundamental to the study of representations over fields of characteristic zero, such as $\mathbb{C}$. This surprising result says that in this situation representations always break apart as direct sums of smaller representations. We need to mention that if we replace $\mathbb{C}$ by different field k, we need $|G|$ to be invertible in k to make sure the following theorem holds.

Theorem 2.1 (Maschke's Theorem) Let W be an invariant subspace of V over a field k such that $|G|$ is invertible in k, then there exists an invariant subspace W' of V such that $V = W \oplus W'$ as representations. In particular, the theorem holds for $k = \mathbb{C}$.

We can give the proof of the theorem using the properties of $\mathbb{C}$, but in this lecture, we will prove the general theorem.

Proof. Since W is a subspace of V, then there exists a complementary subspace W_1 such that $V = W \oplus W_1$ as linear spaces, and take $\pi : V \to W$ be the projection map. Then we get $V = W \oplus \ker \pi$ as linear spaces.

We need to emphasize that this does NOT prove the theorem, since $\ker \pi$ is not necessary to be invariant. Consider the map

$$\pi' := \frac{1}{|G|} \sum_{g \in G} g \pi g^{-1},$$

then π' is linear. Since

$$\pi'(w) = \frac{1}{|G|} \sum_{g \in G} g \pi (g^{-1} w) = w$$

for $w \in W$ and $\pi'(v) \in W$; hence $\pi' : V \twoheadrightarrow W$, then $V / \ker \pi' \simeq W$ or $V = \ker \pi' \oplus W$. Now we need to prove that $\ker \pi'$ is invariant, since $\pi'(hv) = h\pi'(v)$ (need to be carefully verified in the lecture). $\square$

2.2. Irreducible Representations & Simple Modules

Because the next results apply more generally than to group representations, we let A be a ring and consider its modules. A nonzero A −module V is said to be **simple** or **irreducible** if V has no $A-$ submodules other than 0 and V.

Example 2.2 When A is a k −algebra, every module of dimension 1 is simple (from the k −linear subspace structure). We have constructed 3 different $\mathbb{C}[S_3]$ −modules, and they are all simple (need verification).

Note 2.3 Not any module of dimension 1 is simple. Consider $\mathbb{Z}/(m)$ where $m \geq 2$ is not a prime number. Hence $\mathbb{Z}/(m)$ is not simple as $\mathbb{Z}$ −modules but it is generated by 1 and has dimension 1. (Emphasize the difference from linear space!)

We can easily get that a nonzero module is simple if and only if it is **generated** by all its nonzero elements.

A module that is the direct sum of simple submodules is said to be **semisimple** or **completely reducible**. A ring A is called **semisimple** if and only if any A −modules are semisimple.

Example 2.4 Let $C_2 = \{-1, 1\}$ be cyclic of order 2 and consider the representation of $\mathbb{R}^2$ sending -1 to $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. There are just 4 invariant spaces (subrepresentations) and

$$\mathbb{R}^2 = \mathrm{Span}\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \oplus \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$$

is the direct sum of 2 (irreducible) subrepresentations. Hence the representation is **completely reducible**.

Example 2.5 Take $k = \mathbb{F}_p$ and $V = k^2$. Let $G = C_p = \{1, g, \cdots, g^{p-1}\}$ be cyclic group of order p. Hence

$$\rho(g^r) = \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}$$

is a representation with $\text{Span}\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$ as the only 1-deg subrepresentation. So it is impossible to write V as the direct sum of 2 nonzero subrepresentations. (Why? Take a look at **Maschke's Theorem**.)

We will now relate the property of semisimplicity to the property that appears in **Maschke's theorem**, namely that every submodule of a module is a direct summand.

The next result may be an application or interpretation of **Maschke's theorem**. Before the proposition, we need the concept of composition of modules.

- We will cover Chapter 6 of *Commutative Algebra* by Atiyah & Macdonald to introduce the concepr of composition of modules before introducing concept like quotient modules in Chapter 2.

Proposition 2.6 Let A be a ring with 1 and let U be an A −module. T.F.A.E

(1) U can be expressed as a direct sum of finitely many simple A −submodules.

(2) U can be expressed as a sum of finitely many simple A −submodules.

(3) U has finite composition length and has the property that every submodule of U is a direct summand of U.

When these three conditions hold, every submodule of U may also be expressed as the direct sum of finitely many simple modules.

The implication (1) ⇒ (2) is immediate. Now first we will prove (2) ⇒ (3) and we need the following lemma.

Lemma 2.7 Suppose $S_1, S_2, \cdots, S_n$ are simple modules and $U = S_1 + \cdots + S_n$ be an A −module. If V is any submodule of U, then there exists a subset $I = \{i_1, \cdots, i_r\}$ of $\{1, 2, \cdots, n\}$ such that $U = V \oplus S_{i_1} \oplus \cdots \oplus S_{i_n}$. In particular,

- V is a direct summand of U, and
- U is semisimple and be the direct sum of some subsets of the S_i's.

Proof. Choose $I \subset \{1, 2, ..., n\}$ maximal subject to the condition that $W = V \oplus \left(\bigoplus_{i \in I} S_i\right)$ is a direct sum. We show that $W = U$. If not, then there exists $S_j \not\subset W$ for some j. Now $S_j \cap W = 0$, since S_j is simple. Hence $S_j + W = S_j \oplus W$, which is a contradiction. □

Proof of Proposition 2.6. We argue by induction on the composition length of U. □

Corollary 2.8 (Maschke's Theorem, Another Version) Let k be a field in which $|G|$ is invertible. Then every finite-dimensional $k[G]$-module is **semisimple**, i.e. any k-representation of G is completely reducible.

Note 2.9 This result puts us in very good shape if we want to know about the representations of a finite group over a field in which $|G|$ is invertible —— for example any field of characteristic zero, such as $\mathbb{C}$. To obtain a description of all possible finite dimensional representations, we need **only describe the simple ones**, and then arbitrary ones are direct sums of these.

Note 2.10 **Corollary 2.8** is a very strong corollary. Indeed, the kind of algebras that can arise when all modules are semisimple is very restricted. The **Artin-Wedderburn**

Theorem we will present reveals the significance of **Maschke's Theorem**. The theorem will also reveal some nontrivial properties of group algebras and the properties of linear representation itself and help us find all the irreducible representations of G.

Proposition 2.11 A is semisimple as ring if and only if A is semisimple as $A-$modules.

Proof. By **Proposition 2.6**. □

3. Irreducible Decomposition of Representations

3.1. Schur's Lemma & Uniqueness of Decomposition

Possibly the most important single technique in representation theory is to consider endomorphism rings. It is the main technique of this chapter, and we will see it in use throughout the course.The first result is basic and will be used time and time again.

Theorem 3.1 (Schur's Lemma) Let A be a ring with 1 and S_1 and S_2 be simple $A-$modules. Then $\mathrm{Hom}_A(S_1, S_2) = 0$ unless $S_1 \simeq S_2$, in which case the endomorphism ring $\mathrm{End}_A(S_1)$ is a division ring. If A is a finite dimensional algebra over an algebraically closed field k, then every $A-$module endomorphism of S_1 is scalar multiplication, i.e. $\mathrm{End}_A(S_1) \simeq k$.

Proof. Suppose $\theta : S_1 \to S_2$ is a nonzero homomorphism. Then $0 \neq \mathrm{im}\ \theta \subset S_2$, so $\mathrm{im}\ \theta = S_2$ by simplicity of S_2, and we see that θ is surjective. Thus, $\ker\theta \neq S_1$, so $\ker\theta = 0$ by simplicity of S_1, and θ is injective. Therefore, θ is invertible, $S_1 \simeq S_2$, and $\mathrm{End}_A(S_1)$ is a division ring.

If A is a finite-dimensional $k-$algebra and k is algebraically closed, then S_1 is a finite-dimensional $k-$linear space. Let θ be an $A-$module endomorphism of S_1 and let λ be an eigenvalue of θ. Now $(\theta - \lambda I) : S_1 \to S_1$ is a singular endomorphism of $A-$modules, so $\theta - \lambda I = 0$ and $\theta = \lambda I$. □

The next result is the main tool in recovering the structure of an algebra from its representations. We use the notation A^{op} to denote the opposite ring of A.

Lemma 3.2 For any ring A with 1, $\mathrm{End}_A(A) = A^{\mathrm{op}}$.

By **Maschke's Theorem**, let k be a field in which $|G|$ is invertible. Then every finite-dimensional $k[G]$-module is **semisimple**, i.e. any k-representation of G is completely reducible. Take a finite-dimensional $k[G]-$module V, hence we get

$$V = S_1^{\oplus n_1} \oplus \cdots \oplus S_r^{\oplus n_r}$$

be the decomposition of simple submodules of V.

Proposition 3.3 The decomposition of simple submodules is unique.

Proof. By **Schur's Lemma** and the following fact

$$\mathrm{Hom}_A(U_1 \oplus U_2, U_3) = \mathrm{Hom}_A(U_1, U_3) \oplus \mathrm{Hom}_A(U_2, U_3),$$
$$\mathrm{Hom}_A(U_3, U_1 \oplus U_2) = \mathrm{Hom}_A(U_3, U_1) \oplus \mathrm{Hom}_A(U_3, U_2).$$ □

3.2. Artin–Wedderburn's Theorem

Theorem 3.4 (Artin-Wedderburn) Let A be a finite dimensional algebra over a field k with the property that every finite-dimensional module is semisimple. Then A is a direct sum of matrix algebras over division rings. Specifically, if

$$A = S_1^{\oplus n_1} \oplus \cdots \oplus S_r^{\oplus n_r},$$

where the $S_1, \cdots, S_r$ are nonisomorphic simple modules occuring with multiplicities $n_1, \cdots, n_r$ as A −modules, then

$$A = M_{n_1}(D_1) \oplus \cdots \oplus M_{n_r}(D_r),$$

where $D_i = \mathrm{End}_A(S_i)^{\mathrm{op}} = k$ when k is algebraically closed.

Proof. First we observe that if we have a direct sum decomposition

$$U = U_1 \oplus U_2 \oplus \cdots \oplus U_r,$$

of a module U then $\mathrm{End}_A(U)$ is isomorphic to the algebra of $r \times r$ matrices in which the i, j entries lies in $\mathrm{Hom}_A(U_j, U_i)$ (why?).

Since $\mathrm{Hom}_A\left(S_i^{\oplus n_i}, S_j^{\oplus n_j}\right) = 0$ if $i \neq j$ by **Schur's Lemma**, the decomposition of A shows that

$$\mathrm{End}_A(A) = \mathrm{End}_A\left(S_1^{\oplus n_1}\right) \oplus \cdots \oplus \mathrm{End}_A(S_r^{\oplus n_r})$$

and furthermore, $\mathrm{End}_A\left(S_i^{\oplus n_i}\right) = M_{n_i}(D_i^{\mathrm{op}})$. (why?) Putting these pieces together gives the matrix algebra decomposition. □

Corollary 3.5 Let A be a finite-dimensional semisimple algebra over a field k. In any decomposition,

$$A = A = S_1^{\oplus n_1} \oplus \cdots \oplus S_r^{\oplus n_r}$$

where the S_i's are pairwise nonisomorphic simple modules, then

- $S_1, \cdots, S_r$ is a complete set of representatives of the isomorphism classes of simple A −modules.
- $\dim_k S_i = n_i$, and

$$\dim_k A = \sum_{i=1}^{r} n_i^2.$$

Proof. Need to be continued… □

Let us now restate what we have proved specifically in the context of group representations.

Corollary 3.6 Let G be a finite group and k be a field in which $|G|$ is invertible.

- As a ring, $k[G]$ is a direct sum of matrix algebras over division rings.
- Suppose k is algebraically closed. Let $S_1, S_2, \ldots, S)r$ be pairwise non-isomorphic simple $k[G]$ −modules and let $n_i = \dim_k S_i$ be the degree of S_i. Then n_i equals the multiplicity with which S_i is a summand of the regular representation of G, and

$$|G| \geq \sum_{i=1}^{r} n_i^2$$

with equality if and only if $S_1, \ldots, S_r$ is a complete set of representatives of the simple $k[G]$ −modules.

The second part of the result provides a numerical criterion that enables us to say when we have constructed all the simple modules of a group over $\mathbb{C}$, algebraically closed field where $|G|$ is invertible: we will see soon that $\sum n_i^2 = |G|$.

Note 3.7 Suppose k is algebraically closed. Notice that

$$z(k[G]) = \bigoplus_{i=1}^{r} z\left(M_{n_i}(k)\right) = k^r,$$

which implies

$$\dim_k z(k[G]) = r.$$

Proposition 3.8 Let

$$G = \coprod_{i=1}^{s} C_i$$

be the conj. classes of G. Define

$$c_i := \sum_{g \in C_i} g \in k[G]$$

check if $c_i \in z(k[G])$ and prove that $\{c_1, c_2, \cdots, c_s\}$ is a basis of $z(k[G])$.

Corollary 3.9 If k is algebraically closed, then

$$|\{k - \text{irr. rep. of } G\}| = |\{\text{cong. classes of } G\}|.$$

Example 3.10 Determine all the representations of S_3.