Differential Manifold | $\boxed{\text{Seminar 3}}$

Recall In analysis

$$\begin{array}{ccc} f: U & \longrightarrow & V \\ \cap & & \cap \\ \mathbb{R}^n & & \mathbb{R}^m \end{array} \quad \text{smooth map}$$

Fact The lineaization of f at a.

$$\lim_{x \to 0} \frac{|f(x) - f(a) - (df_a)(x-a)|}{|x-a|} = 0.$$

the differential (tangent map)

$$\rightsquigarrow \; df_a : \begin{array}{ccc} \mathbb{R}^n & \longrightarrow & \mathbb{R}^m \\ \parallel & & \parallel \\ T_a U & & T_{f(a)} V \end{array}$$

linear map whose matrix is the Jacobian matrix

$$(df_a) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

d as a functor of POPEN $\rightsquigarrow$ $\mathbb{R}$-linear space

① $d(\mathrm{Id}) = \mathrm{Id}_{T_a U}$ ② $d(g \circ f)_a = dg_{f(a)} \circ df_a$.

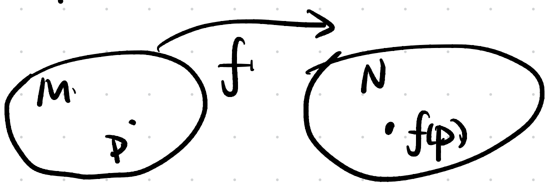
Seminar 3 Jan 24. Differential & Local behavior of sm. maps.

• Differential

• Smooth

Object = smooth manifold + point

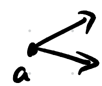
Morphism = smooth maps



$\vec{v} \in \mathbb{R}^m$

$$f: \mathbb{R}^m \longrightarrow \mathbb{R}^n$$
$$\underline{\underline{p}} \longmapsto f(p)$$

$$D^p_{\underline{\underline{\vec{v}}}} f = df_a(\vec{v})$$



$$\underline{\underline{df_p}} : \underline{\underline{\mathbb{R}^m}} \longrightarrow \mathbb{R}^n$$

$$(df_p) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \vdots & & \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_m} \end{pmatrix}.$$

$$\underline{D^a_{\vec{v}}} : \underline{C^\infty(\mathbb{R}^n) \longrightarrow \mathbb{R}}$$

① $D^a_{\vec{v}}$ linear map

② Leibniz equation

$$D^a_{\vec{v}}(fg) = D^a_{\vec{v}}(f)\, g(a) + f(a)\, D^a_{\vec{v}}(g)$$

$$df_p : \underline{\underline{T_p\mathbb{R}^m}} \to T_{f(p)}\mathbb{R}^n \qquad D: C^\infty(\underline{\underline{M}}) \longrightarrow \mathbb{R}$$

Def (tangent space)

- M = n-dim smooth manifold.

Linear map $X_p : \mathcal{C}^\infty(M) \to \mathbb{R}$.

$$X_p(fg) = f(p)X_p(g) + X_p(f)g(p) \quad \forall f, g \in \mathcal{C}^\infty(M)$$

X_p is a "tangent vector" of M at p

- $T_pM = \{X_p\}$ is the tangent space of M at p

Def (differential)

- $f : M \to N$ $\qquad p \in M$.

$$df_p : T_pM \to T_{f(p)}N \text{ linear map}$$
$$X_p \mapsto df_p(X_p)$$

$$df_p(X_p) : \mathcal{C}^\infty(N) \to \mathbb{R}$$
$$g \mapsto X_p(g \circ f)$$

$$(M, p) \xrightarrow{f} (N, f(p)) \underset{d}{\rightsquigarrow} T_pM \xrightarrow{df_p} T_{f(p)}N$$

- $N = \mathbb{R}$ $\quad f \in \mathcal{C}^\infty(M, N) = \mathcal{C}^\infty(M)$

$$df_p : T_pM \to T_{f(p)}\mathbb{R} \quad (*)$$

$$T_{f(p)}\mathbb{R} = \mathbb{R}$$

$$\boxed{df_p(X_p) = X_p(f)}$$

$\boxed{\varphi} \rightsquigarrow GL(n, \mathbb{R})$ $\quad$ det

T_pM $\quad (\varphi, U, V)$ $\quad \varphi : (U, p) \xrightarrow{\boxed{\sim}} (V, f(p))$, $V \subset \mathbb{R}^n$

d functor $\rightsquigarrow$ $\quad d\varphi_p : T_pU \xrightarrow{\boxed{\sim}} T_{f(p)}V$ $\quad \mathbb{R}^n$

$$T_{f(p)}V = \left\langle \boxed{\frac{\partial}{\partial x^1}}, \frac{\partial}{\partial x^2}, \ldots, \frac{\partial}{\partial x^n} \right\rangle \qquad \left.\frac{\partial}{\partial x^1}\right|_{f(p)}$$

$$T_pU = \left\langle (d\varphi_p)^{-1}\left(\frac{\partial}{\partial x^1}\right), \ldots, (d\varphi_p)^{-1}\left(\frac{\partial}{\partial x^n}\right) \right\rangle$$

$$\left.\frac{\partial}{\partial x^i}\right|_p = (d\varphi_p)^{-1}\left(\left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)}\right)$$

$$T_pU = \left\{ v^1 \left.\frac{\partial}{\partial x^1}\right|_p + \cdots + v^n \left.\frac{\partial}{\partial x^n}\right|_p \right\}$$

$$v^1, \ldots, v^n \in \mathbb{R}$$

- $f: M \xrightarrow{\sim} N \rightsquigarrow \underline{df_p : T_pM \xrightarrow{\sim} T_{f(p)}N}$

$df_p : T_pM \xrightarrow{\sim} T_{f(p)}N \rightsquigarrow p \in U_p,\ f(p) \in V_{f(p)}$

$\underline{f|_{U_p} : U_p \xrightarrow{\sim} V_{f(p)}}$

- $f: M \to N$ p local diffeomorphism

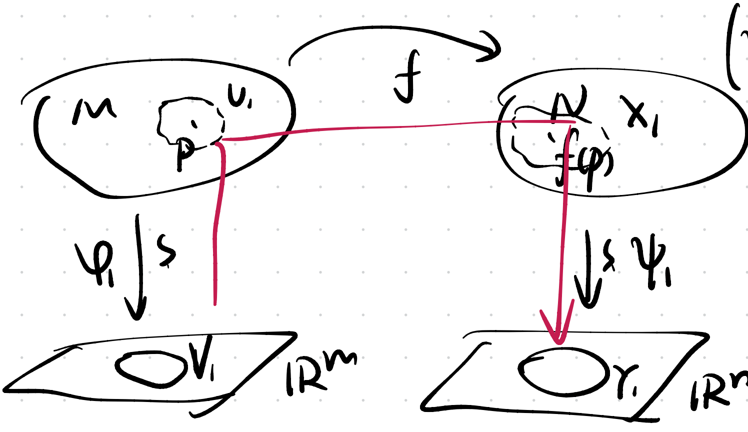
$+$ f bijective $\rightsquigarrow$ f diffeomorphism.

$\underline{f: M \to N}$ $\boxed{p}$ $\underline{df_p : T_pM \to T_{f(p)}N}$

(<u>Rank Theorem</u>)

$f: M \to N$ constant rank r near p

$p \cdot (\varphi_1, U_1, V_1)$ $f(p)$ (ψ_1, X_1, Y_1)



$(\psi_1 \circ f \circ \varphi_1^{-1})(x_1, \dots, x_n)$

$= (x_1, \dots, x_r, 0, \dots, 0)$

- $\underline{r = n} \le m.$ $\quad (x_1, \dots, x_m) \longmapsto (x_1, \dots, x_n)$

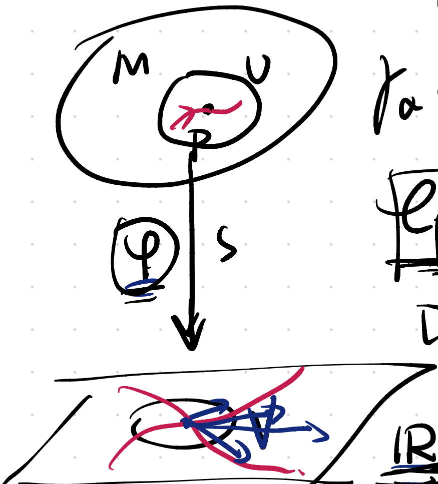
  x_n $\quad$ canonical submersion

- $\underline{r = m} \le n$ $\quad (x_1, \dots, x_m) \longmapsto (\underbrace{x_1, \dots, x_m, 0, \dots, 0}_{n})$

  canonical immersion

2. $\mathcal{C}_p$

$\gamma : (-\varepsilon, \varepsilon) \to U \qquad \gamma(0) = p$



$\gamma_\alpha \sim \gamma_\beta \iff \frac{d(\varphi \circ \gamma_\alpha)}{dt}(0) = \frac{d(\varphi \circ \gamma_\beta)}{dt}(0)$

$\boxed{\mathcal{C}_p / \sim} \xrightarrow{\sim} T_pM$

$[\gamma] \longmapsto X_\gamma$

$\mathbb{R}^n$ $\quad \boxed{X_\gamma : f \longmapsto \frac{d(f \circ \gamma)}{dt}(0)}$

$[\gamma_1] + [\gamma_2] = [\varphi^{-1}(\varphi \circ \gamma_1 + \varphi \circ \gamma_2)]$

$[\gamma_1'] + [\gamma_2'] = [\varphi^{-1}(\varphi \circ \gamma_1 + \varphi \circ \gamma_2)]$

$\lambda[\gamma] = [\varphi^{-1}(\lambda \varphi \circ \gamma)]$

2. • M, N smooth manifolds

$$M\times N = \{(\varphi_\alpha\times\psi_\beta,\ U_\alpha\times X_\beta,\ V_\alpha\times Y_\beta)\}$$

• $\pi_1 : \underline{M\times N} \twoheadrightarrow \underline{M}$ $\quad (\varphi_{\alpha'}, U_{\alpha'}, V_{\alpha'})$

$(\varphi_\alpha\times\psi_\beta,\ U_\alpha\times X_\beta,\ V_\alpha\times Y_\beta)$

$$\varphi_{\alpha'}\circ\pi_1\circ(\varphi_\alpha\times\psi_\beta)^{-1}$$

$$= \varphi_{\alpha'}\circ\pi_1\circ(\varphi_\alpha^{-1}\times\psi_\beta^{-1}) = \varphi_{\alpha'}\varphi_\alpha^{-1} = \varphi_{\alpha\alpha'}$$

•

$$\begin{tikzcd}
M & & \\
& M\times N \arrow[ul, "\pi_1"'] \arrow[dl, "\pi_2"] & P \arrow[l, "f"] \arrow[ull, bend right, "\pi_1\circ f"'] \arrow[dll, bend left, "\pi_2\circ f"] \\
N & &
\end{tikzcd}$$

f smooth $\iff$ $\pi_1\circ f$, $\pi_2\circ f$ smooth

• $T_{(p_1,p_2)}(M_1\times M_2) \simeq T_{p_1}M_1 \oplus T_{p_2}M_2$

$\pi_1 : M_1\times M_2 \twoheadrightarrow M_1$ —

☆ $\Phi : T_{(p_1,p_2)}(M_1\times M_2) \xrightarrow[\sim]{(d\pi_1)_{p_1}\oplus(d\pi_2)_{p_2}} T_{p_1}M_1\oplus T_{p_2}M_2$

$v \longmapsto (d\pi_1(v), d\pi_2(v))$ $\quad v_2$

$M_1 \xhookrightarrow{\iota} M_1\times M_2$

$m_1 \longmapsto (m_1, p_2)$

$$\Phi^{-1}: T_{p_1}M_1 \oplus T_{p_2}M_2 \longrightarrow T_{(p_1,p_2)}(M_1 \times M_2)$$

$$(v_1, v_2) \longmapsto (d\iota_1)_{p_1}(v_1) + (d\iota_2)_{p_2}(v_2)$$

$$(d\iota_1)(d\pi_1(v)) + d(\iota_2(d\pi_2(v)))$$

$\pi_1 \circ \iota_1 = id_{M_1}$ $\quad d\pi_1(d\iota_1(v_1) + d\iota_2(v_2))$

$\pi_2 \circ \iota_2 = id_{M_2}$ $\quad = (d\pi_1\, d\iota_1(v_1) + d\pi_1\, d\iota_2(v_2))$

$\pi_1 \circ \iota_2$ constant $\quad = d(\pi_1 \iota_1)(v_1) \quad d(\pi_1 \iota_2)$

$\pi_2 \circ \iota_1$ constant $\quad = \boxed{v_1}$

- $M_1 \xrightarrow{f_1} N_1$ $\qquad M_1 \times M_2 \xrightarrow{f_1 \times f_2} N_1 \times N_2$

$T_{p_1}M_1 \xrightarrow{df_1} T_{f_1(p_1)}N_1$

$M_2 \xrightarrow{f_2} N_2$

$T_{p_2}M_2 \xrightarrow{df_2} T_{f_2(p_2)}N_2$

$$\begin{array}{ccc} T_{(p_1,p_2)}(M_1 \times M_2) & \xrightarrow{d(f_1 \times f_2)} & T_{\dots}(N_1 \times N_2) \\ \downarrow \wr & & \downarrow \wr \\ T_{p_1}M_1 \oplus T_{p_2}M_2 & \xrightarrow[df_1 \oplus df_2]{} & T_{-}N_1 \oplus T_{-}N_2 \end{array}$$

- tangent bundle.

- M n-dim. $\quad \underline{\underline{TM}} = \underset{\underline{\underline{p \in M}}}{\coprod} \underline{\underline{T_pM}}$

$\quad\quad\quad\quad \cap$

$\quad\quad (\boxed{p}, \boxed{X_p})$

$$\pi : TM \twoheadrightarrow M$$
$$(p, X_p) \mapsto p$$

$$T\varphi = (\underbrace{\varphi \circ \pi}_{n}, \underbrace{d\varphi}_{n}) : \pi^{-1}(U) \to V \times \mathbb{R}^n$$
$$(p, X_p) \mapsto (\underline{\varphi(p)}, \underline{d\varphi_p(X_p)})$$

$$(T\varphi)_\alpha \circ (T\varphi)_\beta^{-1} = (\underline{\varphi_\alpha \circ \varphi_\beta^{-1}}, \underline{(d\varphi_\alpha)_p \circ (d\varphi_\beta)_p^{-1}})$$

$$(d\varphi_\alpha)_p \circ (d\varphi_\beta)_p^{-1} = (d\varphi_\alpha) \circ (d(\varphi_\beta^{-1}))_{\varphi_\beta(p)}$$
$$= \underline{(d(\varphi_\alpha \circ \varphi_\beta^{-1}))_{\varphi_\beta(p)}}$$

- $$\begin{pmatrix} d(\varphi_\alpha \circ \varphi_\beta^{-1})_{\varphi_\beta(p)} & \\ & \underline{d(\varphi_\alpha \circ \varphi_\beta^{-1})_{\varphi_\beta(p)}} \end{pmatrix}$$

$$\det \left(\underline{d(\varphi_{\beta\alpha})_{\varphi_\beta(p)}} \right)^2 > 0 \qquad TM \ \underline{\text{orientable}}$$

$\underline{(d) \quad (e)}$

7. $GL(n, \mathbb{R})$ $\quad A \in GL(n, \mathbb{R})$

$$\underline{\underline{\varphi}}(A) = (\underbrace{a_{11}, a_{12}, \ldots, a_{1n}, a_{21}, \ldots\ldots a_{nn}}_{\delta})$$

- $\det : GL(n, \mathbb{R}) \to \mathbb{R}$

$\underline{\underline{\det}} \circ \underline{\underline{\varphi}}^{-1}$ smooth ✓

- $\underline{\underline{X}} \in GL(n, \mathbb{R})$ $\quad \underline{T_X(GL(n, \mathbb{R}))} \simeq \underline{\mathbb{R}^{n^2}}$ $\quad \boxed{M(n)}$

$$= \left\langle d\varphi^{-1}\left(\frac{\partial}{\partial x_{11}}, \ldots \frac{\partial}{\partial x_{nn}}\right)\right\rangle$$

$$= \left\langle d\varphi^{-1}\left(\frac{\partial}{\partial x_{11}}\right), \ldots, d\varphi^{-1}\left(\frac{\partial}{\partial x_{nn}}\right)\right\rangle$$

$$= \left\langle \frac{\partial}{\partial x_{11}} \ldots \frac{\partial}{x_{nn}} \right\rangle$$

$\det \in C^\infty(GL_n(\mathbb{R}))$

$$(d\,\underline{\underline{\det}})_X : \underline{T_X(GL_n(\mathbb{R}))} \to \mathbb{R} = T_{\det(X)}\mathbb{R}$$

$$\underline{(d\det)_X(\boxed{A})} = A(\det) = \sum_{i,j} a_{ij} \frac{\partial \det}{\partial x_{ij}}\bigg|_X$$

$$A = (a_{ij})_{n\times n} = \underline{\sum_{i,j} a_{ij} \frac{\partial}{\partial x_{ij}}} = \sum_{i,j} a_{ij} \underline{X_{ij}}$$

$X^{-1} \cdot X = I$

$X \cdot X^* = \det X$

$$= \boxed{\det X \; \mathrm{tr}(X^{-1}A)}$$

$$= \mathrm{tr}(X^* A)$$

$\underline{\det}$ $\quad (d\det)_X : \boxed{\mathbb{R}^{n^2}} \longrightarrow \boxed{\mathbb{R}}$ Submersion

$\boxed{A}$

(b). $\underline{f}$: $GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$ $\quad X \mapsto \underline{X^TX}$

$\cap \qquad \cap$

$\underline{\mathbb{R}^{n^2}} \qquad \underline{\mathbb{R}^{n^2}}$

$$f(X+A) - f(X) = f_*(\underline{A}) + \underline{o(A)}^{\mathbb{R}^{n^2}}$$

$$(X^T + A^T)(X+A) - XX^T = \underline{A^TX + X^TA} + \underline{A^TA}$$

$$\underline{df_X} : \mathbb{R}^{n^2} \to \mathbb{R}^{n^2} \qquad \frac{n(n+1)}{2}$$

$$A \mapsto \underline{A^TX + X^TA} \quad \checkmark$$

$$B \rightsquigarrow \frac{B}{2} \qquad \frac{B}{2} \quad \frac{B}{2} \qquad \text{rank } df_X = \frac{n(n+1)}{2}$$

$$\frac{B}{2} = A^T \underline{X} \qquad \underline{A^T} = \frac{1}{2} B X^{-1}$$